# Inverse Trigonometric Function

## Domain and Range of Inverse Trigonometric Functions :-

| $f(x)$ | Domain | Range |
|---|---|---|
| $\sin^{-1}x$ | $x \in [-1,1]$ or $\lvert x\rvert \leq 1$ | $[-\pi/2, \pi/2]$ |
| $\cos^{-1}x$ | $x \in [-1,1]$ or $\lvert x\rvert \leq 1$ | $[0, \pi]$ |
| $\tan^{-1}x$ | $x \in R$ | $(-\pi/2, \pi/2)$ |
| $\cot^{-1}x$ | $x \in R$ | $(0, \pi)$ |
| $\sec^{-1}x$ | $\lvert x\rvert \geq 1$ or $x \in R - (-1,1)$ | $[0, \pi] - \{\pi/2\}$ |
| $\text{cosec}^{-1}x$ | $\lvert x\rvert \geq 1$ or $x \in (-\infty,-1] \cup [1,\infty)$ | $[-\pi/2, \pi/2] - \{0\}$ |

## Graph of Inverse Trigonometric Functions :-

1. $\sin^{-1}x$

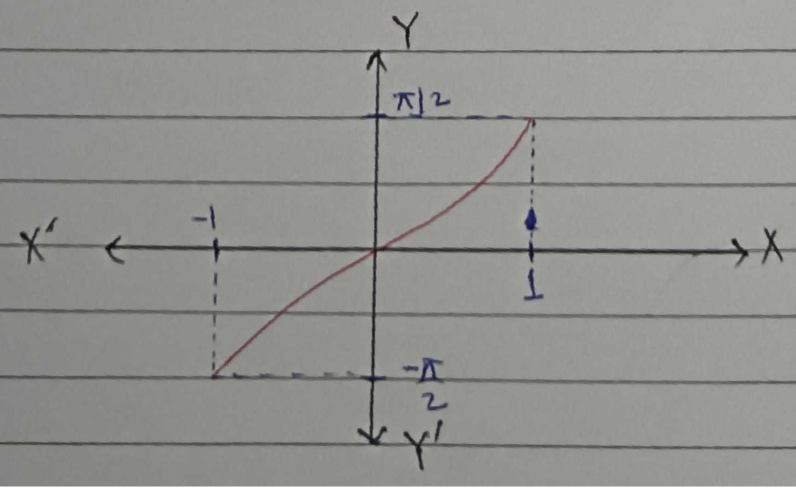
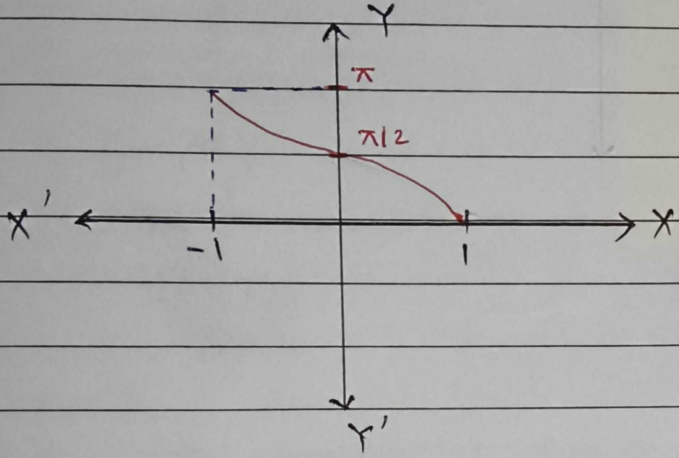



Image of $y = \sin x$ about $y = x$

Domain:- $x \in [-1, 1]$

Range:- $[-\pi/2, \pi/2]$

2. $\cos^{-1}x$



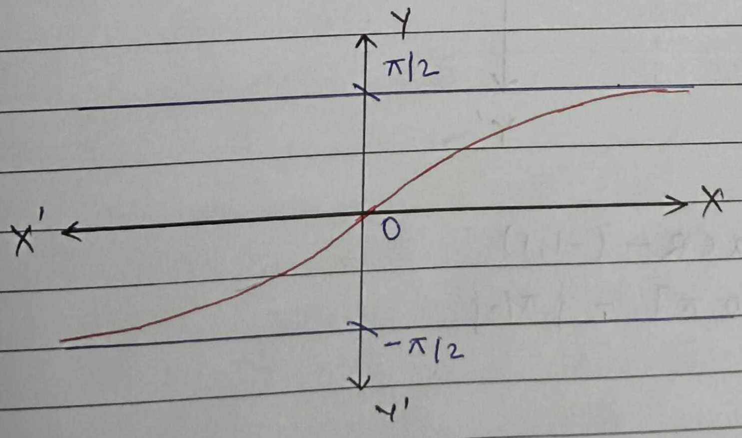

$-1 < x < \frac{1}{\sqrt{2}}$, $\cos^{-1}x > \sin^{-1}x$

$\frac{1}{\sqrt{2}} < x \leq 1$, $\sin^{-1}x > \cos^{-1}x$

Domain:- $[-1, 1]$

Range:- $[0, \pi]$

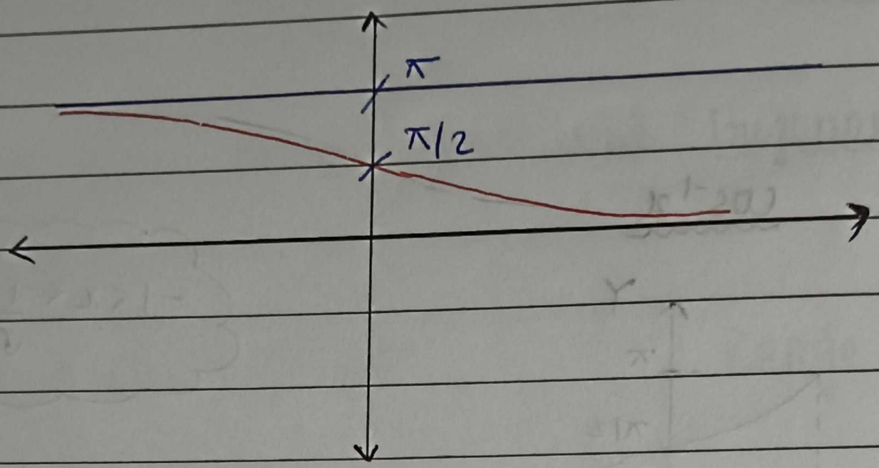
3. $\tan^{-1}x$




Domain:- $x \in R$

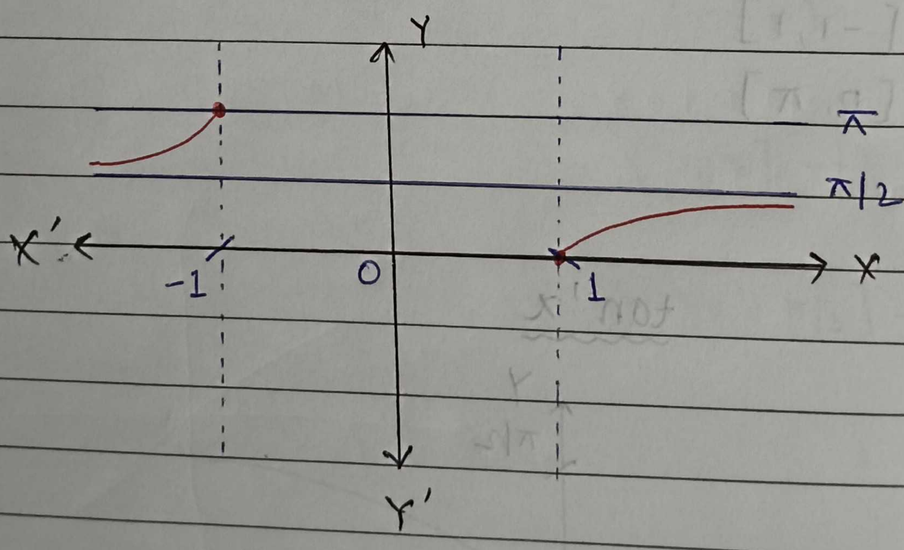
Range:- $(-\pi/2, \pi/2)$

4. $\cot^{-1}x$



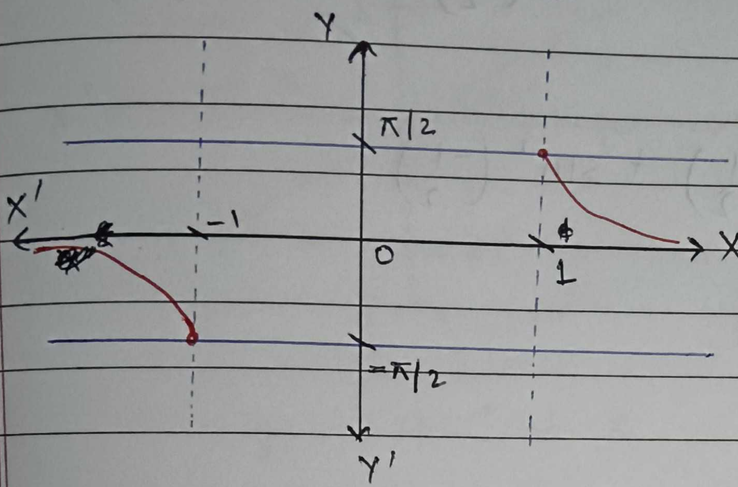

Domain :- $x \in R$
Range :- $(0, \pi)$

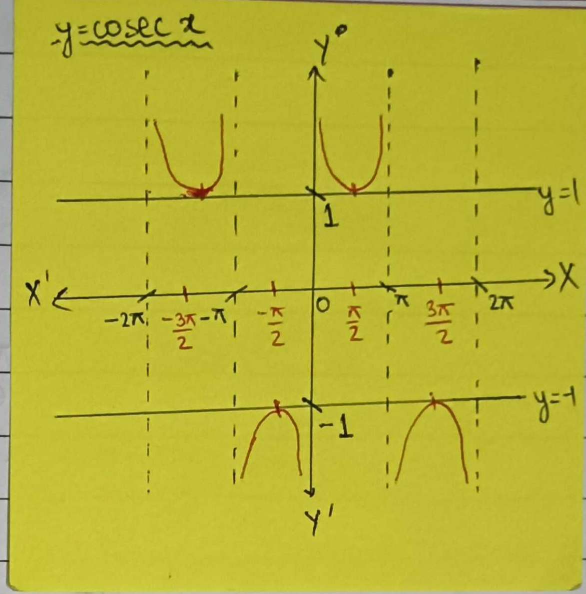
5. $\sec^{-1}x$




Domain :- $x \in R - (-1, 1)$
Range :- $[0, \pi] - \{\pi/2\}$

6. $\underline{\cosec^{-1}x}$




Note :

1. All the Inverse Trigonometric functions represent an angle.

2. If $x > 0$, then all six ITF i.e., $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\cosec^{-1}x$, represent an acute angle.

3. If $x < 0$, then $\sin^{-1}x$, $\tan^{-1}x$ and $\cosec^{-1}x$ represent an angle from $-\pi/2$ to $0$. ($4^{th}$ quadrant).

4. If $x < 0$, then $\cos^{-1}x$, $\cot^{-1}x$ and $\sec^{-1}x$ represent an obtuse angle ($2^{nd}$ quandrant).

Third quadrant is never used in range of ITF.

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Q. The value of $\tan^{-1}1 + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$ is.

Soln:- $\tan^{-1}1 + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12}$$

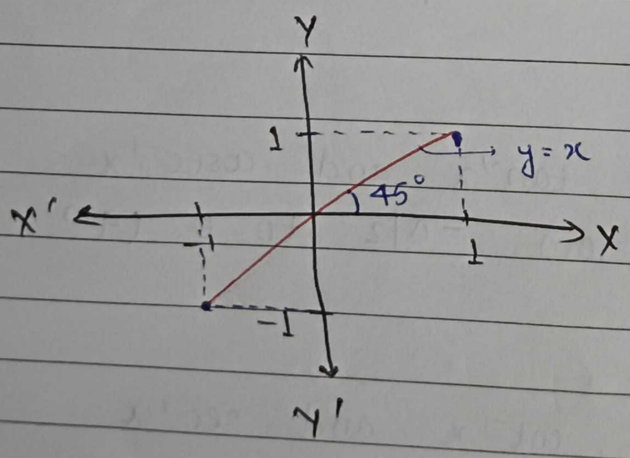
$$= \frac{9\pi}{12} = \frac{3\pi}{4}$$

## Properties of Inverse Circular functions:-

Property 1

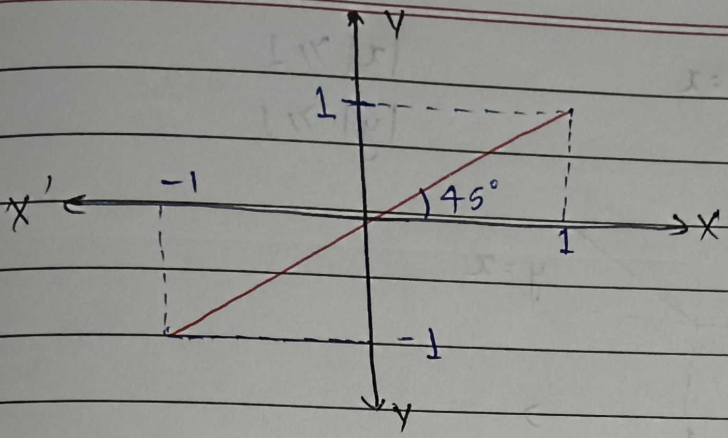
1. $y = \sin(\sin^{-1}x) = x$ , $x \in [-1, 1]$

$y \in [-1, 1]$



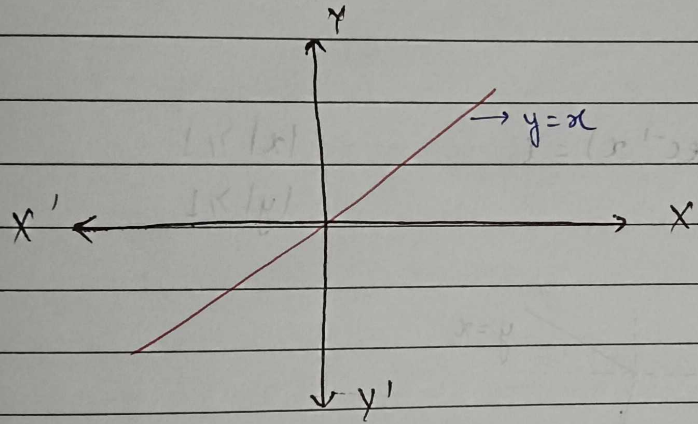

2. $y = \cos(\cos^{-1}x) = x$ , $x \in [-1, 1]$

$y \in [-1, 1]$



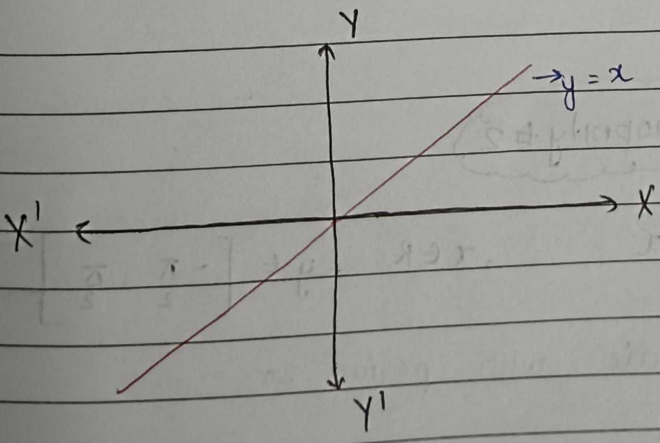

3. $y = \tan(\tan^{-1}x) = x$ $\quad x \in R$

$y \in R$



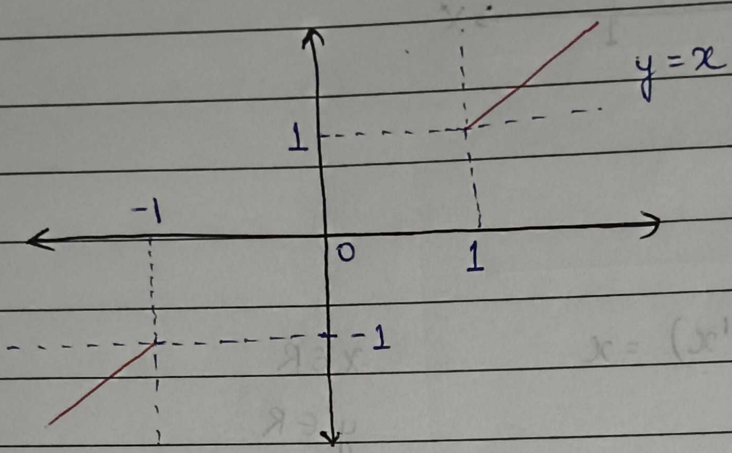

4. $y = \cot(\cot^{-1}x) = x$ $\quad x \in R$

$y \in R$



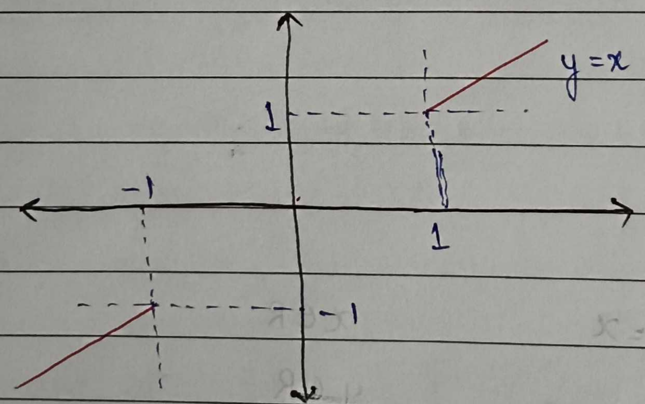
5. $y = \sec(\sec^{-1} x) = x$ $\qquad |x| \geq 1$

$|y| \geq 1$




6. $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$ $\qquad |x| \geq 1$

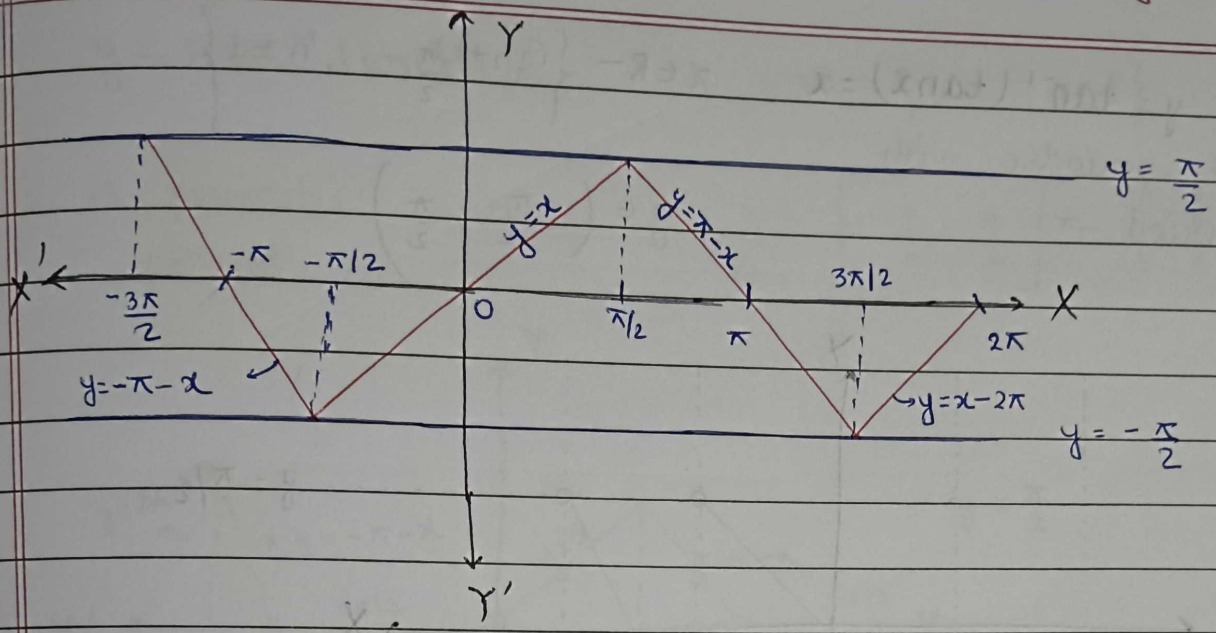
$|y| \geq 1$




## Property # 2

1. $y = \sin^{-1}(\sin x) = x \quad , x \in R \quad , y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

is periodic with period $2\pi$.



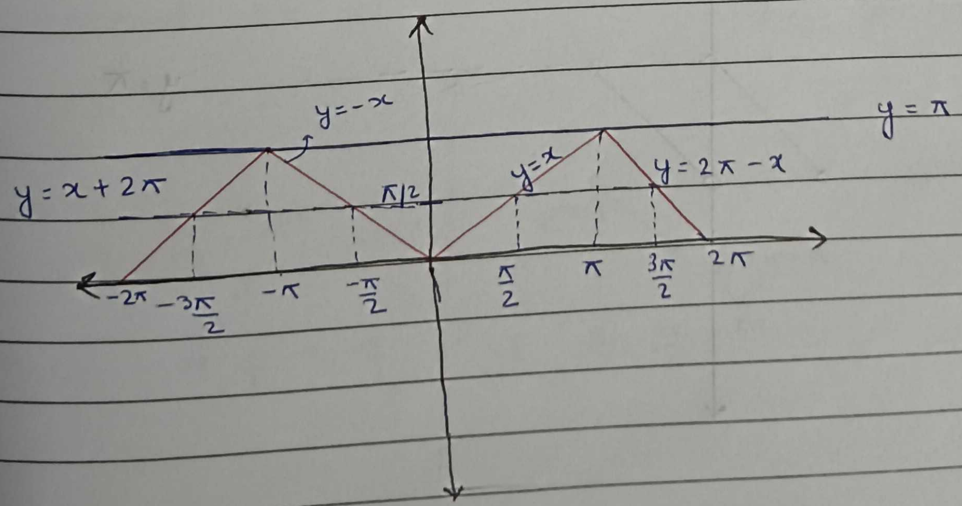

eg. $\sin^{-1}(\sin 5)$ → radian

B/w, $\frac{3\pi}{2}$ ~~to~~ and $2\pi$

$\sin^{-1}(\sin 5) = 5 - 2\pi$

2. $y = \cos^{-1}(\cos x) = x$ , $x \in R$ , $y \in [0, \pi]$

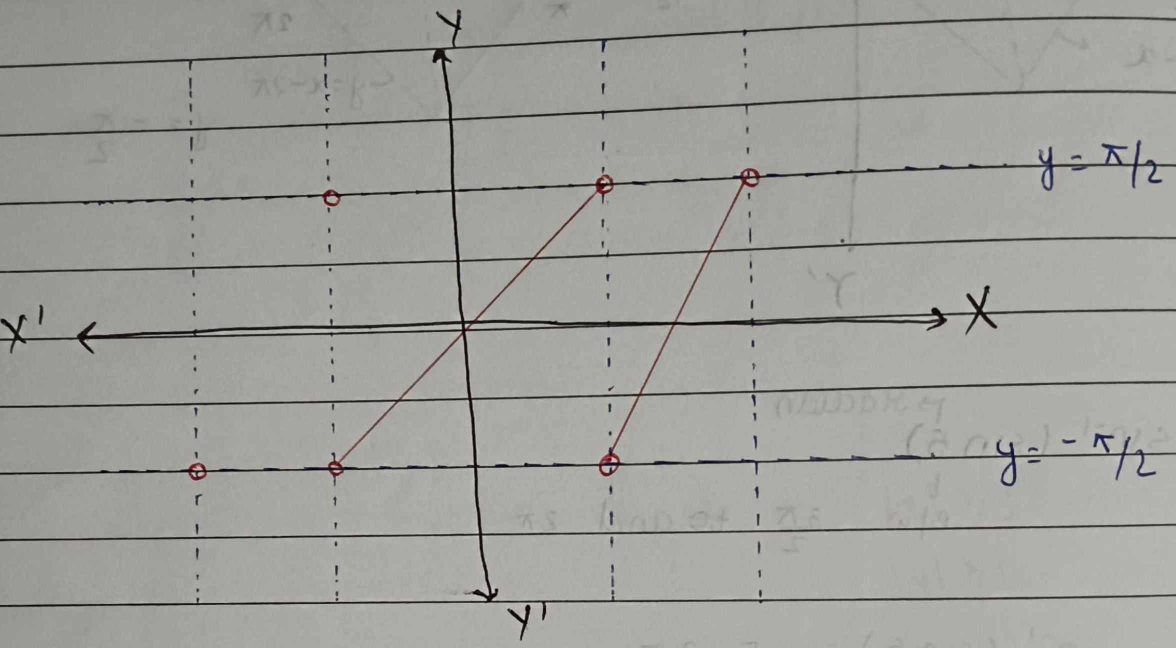
is periodic with period $2\pi$.



3. $y = \tan^{-1}(\tan x) = x \quad x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

is periodic with period $\pi$

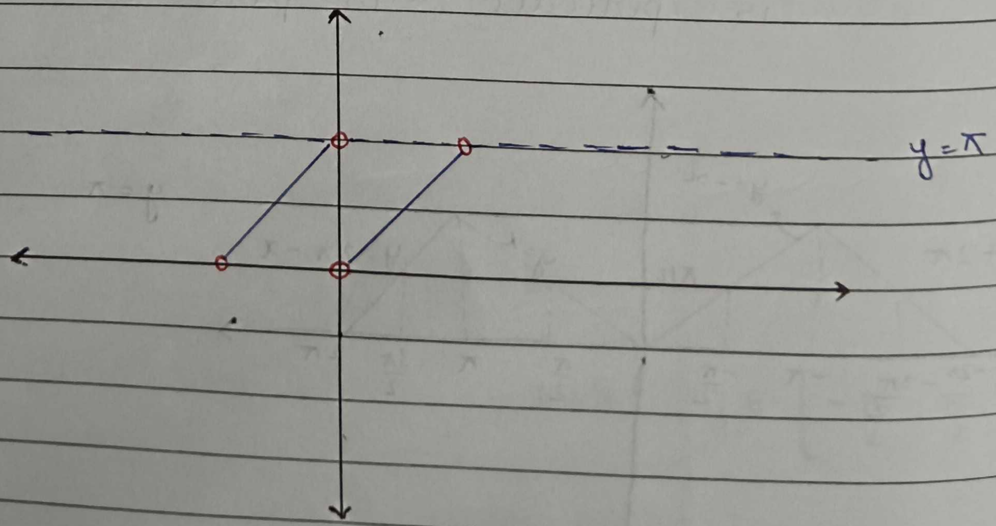
$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$




4. $y = \cot^{-1}(\cot x) = x \quad x \in R - \{n\pi, n \in I\}$

$y \in (0, \pi)$

is periodic with period $\pi$



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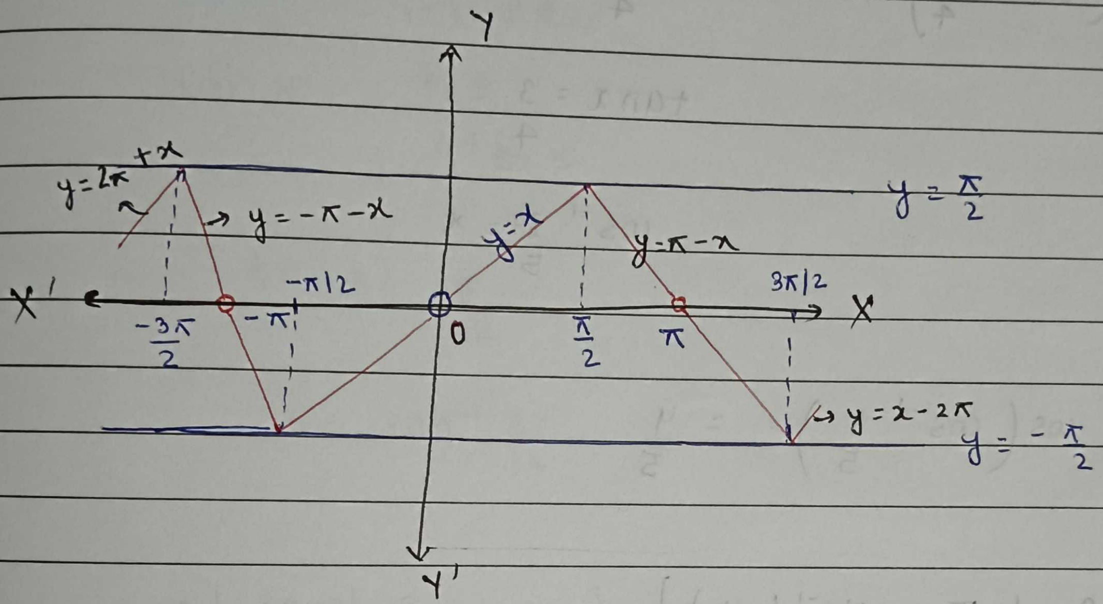
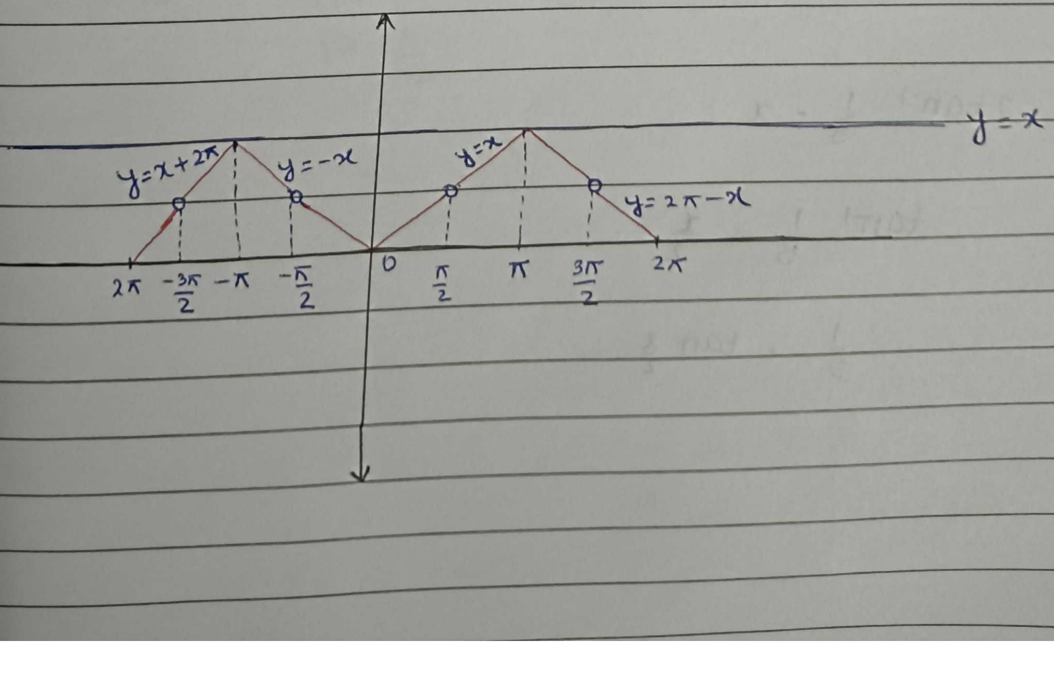
5. $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$ $\quad x \in R - \{n\pi, n \in I\}$

is periodic with period $2\pi$. $\quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$




6. $y = \sec^{-1}(\sec x) = x$ $\quad , x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

is periodic with period $2\pi$. $\quad y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$



Q. $\sin\left(\cos^{-1}\frac{3}{5}\right) \Rightarrow \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}$

Q. $\cos\left(\tan^{-1}\frac{3}{4}\right) \Rightarrow \tan^{-1}\frac{3}{4} = x$

$\tan x = \frac{3}{4}$

$\cos^{-1}\frac{\cancel{3}4}{5} = x$

$\cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}$

Q. $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Sol$^n$:- $\cancel{\sin\left(\frac{\pi}{2} - \sin^{-1}\right.}$ $\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Q. $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$

Sol$^n$:- $2\tan^{-1}\frac{1}{5} = x$

$\tan^{-1}\frac{1}{5} = \frac{x}{2}$

$\frac{1}{5} = \tan\frac{x}{2}$

$$\therefore \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\because \tan x = \frac{2\tan x/2}{1-\tan^2 x}$$

$$\tan x = \frac{\cancel{2}/5}{\cancel{24}^{12}/\cancel{25}\,5}$$

$$\tan x = \frac{5}{12}$$

$$x = \tan^{-1}\frac{5}{12}$$

$$\tan\left(\underbrace{\tan^{-1}\frac{5}{12}}_{A} - \underbrace{\frac{\pi}{4}}_{B}\right)$$

$$= \frac{\tan(\tan^{-1}5/12) - \tan \pi/4)}{1+ \tan(\tan^{-1}5/12)\ \tan \pi/4}$$

$$= \frac{5/12 - 1}{1+\frac{5}{12}} = \frac{-7}{17}$$

## Formulae :-

P-1 :-

1. $\sin(\sin^{-1}x) = x$ , $|x| \leq 1$ $\xrightarrow{or}$ $x \in [-1, 1]$

2. $\cos(\cos^{-1}x) = x$ , $|x| \leq 1$

3. $\tan(\tan^{-1}x) = x$ , $x \in R$

4. $\cot(\cot^{-1}x) = x$ , $x \in R$

P-2 :

1. $\sin^{-1}(\sin x) = x$ , $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

2. ~~sin~~ $\cos^{-1}(\cos x) = x$ , $x \in [0, \pi]$

3. $\tan^{-1}(\tan x) = x$ , $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

P-3 :-

1. $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ , $|x| \geq 1$

2. $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$ , $|x| \geq 1$

3. $$\cot^{-1}x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$$

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## P-4 :-

1. $\sin^{-1}(-x) = -\sin^{-1}x$ , $\forall |x| \leq 1$
2. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$ , $|x| \geq 1$
3. $\tan^{-1}(-x) = -\tan^{-1}x$ , $\forall x \in R$
4. $\cot^{-1}(-x) = \pi - \cot^{-1}x$ , $x \in R$
5. $\sec^{-1}(-x) = \pi - \sec^{-1}x$ , $|x| \geq 1$
6. $\cos\cancel{ec}^{-1}(-x) = \pi - \cos^{-1}x$ , $|x| \leq 1$

## P-5 :-

1. $\sin^{-1}x + \cos^{-1}x = \pi/2$ , $|x| \leq 1$
2. $\tan^{-1}x + \cot^{-1}x = \pi/2$ , $x \in R$
3. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$ , $|x| \geq 1$

## P-6 :-

1. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ , $\left\{ \begin{matrix} x>0, y>0 \\ xy<1 \end{matrix} \right\}$

$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ $\left\{ \begin{matrix} x>0, y>0 \\ xy>1 \end{matrix} \right\}$

2. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ $\left\{ x>0, y>0 \right\}$

## P-7 :-

1. $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ , $x \geq 0$, $y \geq 0$ and $x^2+y^2 \leq 1$

1. $\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \quad \{x \geq 0, y \geq 0, x^2+y^2 \geq 1\}$

2. $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \quad \{x \geq 0, y \geq 0\}$

3. $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \quad \{x \geq 0, y \geq 0\}$

$\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \quad \{$

If $x < y$, $\cancel{xy \geq 0}$ $x > 0$, $y > 0$

4. $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \quad \{x < y, x > 0, y > 0\}$

$= -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) \quad \{x > y, x > 0, y > 0\}$

P-8:-

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-xz}\right)$,

$x > 0, y > 0, z > 0$ and $xy+yz+zx < 1$

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then
$x+y+z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, then
$xy + yz + xz = 1$

P-9 :-

* $2\tan^{-1}x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

1. $$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & |x| \leq 1 \\ \pi - 2\tan^{-1}x, & x > 1 \\ -(\pi + 2\tan^{-1}x), & x < -1 \end{cases}$$

2. $$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & x \geq 0 \\ -2\tan^{-1}x, & x < 0 \end{cases}$$

3. $$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}x, & |x| < 1 \\ \pi + 2\tan^{-1}x, & x < -1 \\ -(\pi - 2\tan^{-1}x), & x > 1 \end{cases}$$

P-10 :-

$$\sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x), & -1 \leq x \leq -1/2 \\ 3\sin^{-1}x, & -1/2 \leq x \leq 1/2 \\ \pi - 3\sin^{-1}x, & 1/2 \leq x \leq 1 \end{cases}$$

$$\cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \leq x \leq -1/2 \\ 2\pi - 3\cos^{-1}x, & -1/2 \leq x \leq 1/2 \\ 3\cos^{-1}x, & 1/2 \leq x \leq 1 \end{cases}$$

$$\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x, & 0 \leq x \leq 1 \\ 2\pi - 2\cos^{-1}x, & -1 \leq x \leq 0 \end{cases}$$

Q The value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is.

Solⁿ:: $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$

$-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\frac{5\pi}{6}\right)$

$-\frac{\pi}{3} + \frac{5\pi}{6} = \frac{-2\pi+5\pi}{6} = \frac{\pi}{2}$

Q. (i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$

(iii) $\cot^{-1}(\cot 4)$

Solⁿ:: (i) $\frac{5\pi}{2} < 10 < \frac{7\pi}{2}$

For odd multiples of $\pi$, $m = -1$

$y = -x + 3\pi$

$y = 3\pi - 10$

(ii) $\tan^{-1}(\tan(-6))$

$-\frac{5\pi}{2} < -6 < -\frac{3\pi}{2}$

For tan, ~~Even multiple of~~ $\pi$,

$m = 1$

$y = x + 2\pi$

$y = 2\pi - 6$

(iii) $\cot^{-1}(\cot 4)$

$\pi < 4 < 2\pi$

odd multiple of $\pi$

$m = 1$

$y = x - \pi$

$y = 4 - \pi$

Q. The value of $\cos^{-1}(\cos 10)$ is

Sol$^n$: Y-coordinate is zero @ even multiples of $\pi$

$3\pi < 10 < 4\pi$ $\quad\quad m = -1$

$y = -x + 4\pi$

$y = 4\pi - 10$

Q. The value of (i) $\sin^{-1}(\sin 8)$ (ii) $\cos^{-1}(\cos 8)$ is.

Sol$^n$:- New method (Short cut Method) for solving these problems:-

First check nearest multiple of $\pi$ which is nearest to given number, such that

$n\pi \pm x \in [-1.57, 1.57]$ {for sin and tan}

[$n\pi \pm x \in$ ~~Domain~~ Range of Trigonometric Ratio]

eg, $8 \rightarrow 8 - 2\pi \quad 8 - 3\pi$

$1.72 \quad -1.42$ ✓

Two possibility : $(8 - 3\pi)$ and $(3\pi - 8)$.

both will give same answer.

Verification:-

$\sin^{-1}(\sin[3\pi + \overbrace{(8 - 3\pi)}^{A}])$

$= -A$

$= 3\pi - 8$

$\sin^{-1}(\sin(3\pi - \overbrace{(3\pi - 8)}^{B}))$

$= B$

$= 3\pi - 8$

(ii) $\cos^{-1}(\cos 8)$

Solⁿ:- $n\pi \pm x \in [0, \pi]$

$n\pi \pm x \in [0, 3.14]$

$8 \to 8 - 2\pi$ $\quad$ 1.72 ✓

$3\pi - 8$ $\quad$ 1.42 ✓

$\cos^{-1}(\cos[2\pi + (8-2\pi)])$

$= -2\pi + 8$

$\cos^{-1}(\cos[3\pi - (3\pi - 8)])$

$= \pi - (3\pi - 8)$

$= -2\pi + 8$

Q. (i) $\cos^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$ (ii) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$

(iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

Solⁿ:- $n\pi \pm x \in [0, \pi]$

$\frac{13\pi}{6} \to \frac{13\pi}{6} - 2\pi$

$\cos^{-1}\left(\cos\left[2\pi + \left(\frac{13\pi}{6} - 2\pi\right)\right]\right)$

$= -2\pi + \frac{13\pi}{6}$

$= \frac{\pi}{6}$

(ii) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$

$n\pi \pm x \in \left(-\pi/2 \ , \pi/2\right)$

$\frac{7\pi}{6} \rightarrow \frac{7\pi}{6} - \pi$

$= \tan^{-1}\left(\tan\left[\pi + \left(\frac{7\pi}{6} - \pi\right)\right]\right)$

$= \frac{7\pi}{6} - \pi = \frac{\pi}{6}$

(iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

$n\pi \pm x \in \left[-\pi/2 \ , \pi/2\right]$

$\frac{5\pi}{6} \longrightarrow \frac{5\pi}{6} - \pi$

$\sin^{-1}\left(\sin\left[\pi + \left(\frac{5\pi}{6} - \pi\right)\right]\right)$

$\cancel{\sin} = - \left(\frac{5\pi}{6} - \pi\right)$

$= \frac{\pi}{6}$

Q. Prove that $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

Soln:- LHS $\Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ $\{xy < 1\}$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2\cancel{0}}{\cancel{91}}}{\frac{9\cancel{0}}{\cancel{91}}}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

LHS = RHS.

Q. Compute $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

Soln:- $\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{35}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{24}}\right)$

$$\tan^{-1}\left(\frac{\frac{\cancel{12}\,6}{\cancel{35}}}{\frac{\cancel{34}\,17}{\cancel{35}}}\right) + \tan^{-1}\left(\frac{\frac{11}{\cancel{24}}}{\frac{23}{\cancel{24}}}\right)$$

$$\tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{66}{17\times23}}\right) = \tan^{-1}\left(\frac{\frac{138+187}{\cancel{391}}}{\frac{391-66}{\cancel{391}}}\right)$$

$= \tan^{-1}\left(\frac{325}{325}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

Q. Compute $\sin^{-1}\left(\frac{12}{13}\right) + \cot^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{63}{16}\right) = ?$

Sol$^n$:- $\cot^{-1}\left(\frac{4}{3}\right) = x$

$\cot x = \frac{4 \to B}{3 \to P}$ $\qquad \sin x = \frac{3}{5}$

$H = 5$

$\sin^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$xy > 1 \leftarrow$

$= \left[\pi + \tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{36}{20}}\right)\right] + \tan^{-1}\left(\frac{63}{16}\right)$

$= \left[\pi + \tan^{-1}\left(-\frac{63}{16}\right)\right] + \tan^{-1}\left(\frac{63}{16}\right)$

$= \pi - \tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \pi$

Q. Prove that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Sol$^n$:- LHS $= \cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{9}{25}} + \frac{3}{5}\sqrt{1-\frac{25}{169}}\right]$$

$$= \sin^{-1}\left[\frac{5}{13} \times \frac{\cancel{16}4}{5} + \frac{3}{5} \times \frac{12}{13}\right]$$

$$= \sin^{-1}\left(\frac{56}{65}\right)$$

Trick to get Triplets

For odd:-

First find square of that number, then break the square into two consecutive terms:

eg, $3 \xrightarrow{3^2} 9 \begin{cases} 4 \\ 5 \end{cases}$

$5 \xrightarrow{5^2} 25 \begin{cases} 12 \\ 13 \end{cases}$

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For even

Find square.
Divide square term by 2.
split into two terms such that difference is 2.

$$8 \xrightarrow{8^2} 64 \xrightarrow{/2} 32 \begin{cases} 15 \\ 17 \end{cases}$$

$$6 \xrightarrow{6^2} 36 \xrightarrow{/2} 18 \begin{cases} 8 \\ 10 \end{cases}$$

$$10 \xrightarrow{10^2} 100 \xrightarrow{/2} 50 \begin{cases} 24 \\ 26 \end{cases}$$

Q $\sin^{-1}\left(\sin \frac{2\pi}{3}\right) + \cos^{-1}\left(\cos \frac{7\pi}{6}\right) + \tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

[JEE Main 2022]

Soln:- $\sin^{-1} \sin\left(\pi - \frac{\cancel{2}\pi}{3}\right) + \cos^{-1}\left(\cos 2\pi - \frac{\cancel{4}5\pi}{6}\right) + \tan^{-1}\tan\left(\pi - \frac{\pi}{4}\right)$

$\sin^{-1} \cancel{\frac{\sqrt{3}}{2}} \sin\left(\frac{\pi}{3}\right) + \cos^{-1}\cos\left(\frac{5\pi}{6}\right) + \tan^{-1}\tan\left(\frac{\pi}{4}\right)$

$\frac{\pi}{3} + \frac{5\pi}{6} \cancel{+} - \frac{\pi}{4}$

$\frac{4\pi + 10\pi \cancel{+} - 3\pi}{12} = \cancel{\frac{17\pi}{12}} \frac{11\pi}{12}$

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Q. $2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ [JEE Main 2020]

Sol$^n$:- $2\pi - \left(\sin^{-1}\left(\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right) + \sin^{-1}\frac{16}{65}\right)$

$2\pi - \left(\sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\frac{16}{65}\right)$

$2\pi - \sin^{-1}\left(\frac{63}{65}\sqrt{1-256}\right.$

$2\pi - \sin^{-1}\left(\frac{63}{65}\times\frac{16}{65} + \frac{16}{65}\times\frac{63}{65}\right)$

$2\pi -$

Sol$^n$:- $2\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{16}{63}\right)$

$2\pi - \left(\tan^{-1}\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{20}{36}} + \tan^{-1}\frac{16}{63}\right)$

$2\pi - \left(\tan^{-1}\frac{\frac{63}{36}}{\frac{16}{36}} + \tan^{-1}\frac{16}{63}\right)$

$2\pi - \left(\tan^{-1}\frac{63}{16} + \tan^{-1}\frac{16}{63}\right)$

$= 2\pi - \left(\cot^{-1}\frac{16}{63} + \tan^{-1}\frac{16}{63}\right)$

$= 2\pi - \frac{\pi}{2}$

$= \frac{3\pi}{2}$

Q. Considering only the principal values of the ITF, the domain of the function

$f(x) = \cos^{-1}\left(\frac{x^2-4x+2}{x^2+3}\right)$ is.

[JEE Main 2022]

Sol$^n$:- $-1 \leq \frac{x^2-4x+2}{x^2+3} \leq 1$

$-x^2-3 \leq x^2-4x+2 \leq x^2+3$

$x^2-4x+2 \geq -x^2-3$

$2x^2-4x+5 \geq 0$

$a>0$, $D<0$

$x \in R$

$x^2-4x+2 \leq x^2+3$

$-4x \leq 1$

$x \geq -\frac{1}{4}$

$x \in \left[-\frac{1}{4}, \infty\right)$

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Q. $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2 = ?$

Soln:- $\cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$$\cos^{-1}\left(\frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2}\right) = \alpha$$

$$\cos\alpha = \frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2}$$

$$\frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2} = \cos\alpha - \frac{xy}{2}$$

Squaring both sides

$$\frac{(1-x^2)(4-y^2)}{4} = \cos^2\alpha + \frac{x^2y^2}{4} - xy\cos\alpha$$

$$(1-x^2)(4-y^2) = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$$

$$4 - y^2 - 4x^2 + \cancel{x^2y^2} = 4\cos^2\alpha + \cancel{x^2y^2} - 4xy\cos\alpha$$

$$4x^2 - 4xy\cos\alpha + y^2 = 4(1-\cos^2\alpha)$$

$$= 4\sin^2\alpha$$

Q. If $\alpha, \beta$ are the roots of the eqn $x^2 - 4x + 1 = 0$ then value of

$f(\alpha,\beta) = \frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$ is-

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Sol$^n$: $x^2-4x+1=0$

$\alpha+\beta=4$

$\alpha\beta=1$

$\alpha = 2+\sqrt{3}$

$\beta = 2-\sqrt{3}$

$\alpha-\beta = 2\sqrt{3}$

$$\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$$

$$=\frac{\beta^3}{2\sin^2\left(\frac{1}{2}\tan^{-1}\beta/\alpha\right)}+\frac{\alpha^3}{2\cos^2\left(\frac{1}{2}\tan^{-1}\alpha/\beta\right)}$$

$$=\frac{\beta^2}{1-\cos\left(\tan^{-1}\beta/\alpha\right)}+\frac{\alpha^3}{1+\cos\left(\tan^{-1}\alpha/\beta\right)}$$

$$=\frac{\beta^3}{1-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}+\frac{\alpha^3}{1+\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha}+\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right)$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\beta^3\left(\sqrt{\alpha^2+\beta^2}\right)+\beta^4+\alpha^3\left(\sqrt{\alpha^2+\beta^2}\right)-\alpha^4}{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\left(\sqrt{\alpha^2+\beta^2}+\beta\right)}\right)$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\sqrt{\alpha^2+\beta^2}\left(\beta^3+\alpha^3\right)+\beta^4-\alpha^4}{\left(\alpha^2+\beta^2\right)-\sqrt{\alpha^2+\beta^2}\left(\alpha-\beta\right)-\alpha\beta}\right)$$

$$=\sqrt{14}\left(\frac{\sqrt{14}\times 52-112\sqrt{3}}{14-\sqrt{14}\cdot 2\sqrt{3}-1}\right)$$

$$=\frac{14\times 52-112\sqrt{42}}{13-2\sqrt{42}}$$

$$= 56\frac{(13-2\sqrt{42})}{(13-2\sqrt{42})}$$

$$= 56$$

Q. $\cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21 + \cot^{-1} 31 + \dots$ n terms.

Soln:. $\underbrace{7 + 13}_{6} + \underbrace{13 + 21}_{8} + \underbrace{21 + 31}_{10} + \dots$

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \tan^{-1}\left(\frac{1}{31}\right) + \dots$$

$$\tan^{-1}\left(\frac{3-2}{1+2\times3}\right) + \tan^{-1}\left(\frac{4-3}{1+4\times3}\right) + \tan^{-1}\left(\frac{5-4}{1+5\times4}\right) + \dots$$

$$= \cancel{\tan^{-1}3} - \tan^{-1}2 + \cancel{\tan^{-1}4} - \cancel{\tan^{-1}3} + \dots + \tan^{-1}(n+2) - \cancel{\tan^{-1}(n+1)}$$

$$\tan^{-1}(n+2) - \tan^{-1}2$$

$$= \tan^{-1}\left(\frac{n}{1+2n+4}\right)$$

$$= \tan^{-1}\left(\frac{n}{2n+5}\right)$$

Q. $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{9}\right) + \dots + \tan^{-1}\left(\frac{2^{n-1}}{1+2^{2n-1}}\right)$

~~upto~~ upto infinity.

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Sol$^n$. $\tan^{-1}\left(\frac{2-1}{1+1\times 2}\right) + \tan^{-1}\left(\frac{4-2}{1+8}\right) + \dots + \tan^{-1}\left(\frac{2^n-2^{n-1}}{1+2^n\cdot 2^{n-1}}\right)$

$$T_n = \tan^{-1}\left(\frac{2^n-2^{n-1}}{1+2^n\cdot 2^{n-1}}\right)$$

$\cancel{\tan^{-1}2} - \tan^{-1}1 + \cancel{\tan^{-1}2^2} - \cancel{\tan^{-1}2} + \dots + \tan^{-1}2^n - \cancel{\tan^{-1}2^{n-1}}$

$\tan^{-1}2^n - \tan^{-1}1$

$n \to \infty$

$$\tan^{-1}\infty - \tan^{-1}1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Q. $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{65}} + \sin^{-1}\frac{1}{\sqrt{325}} + \dots + \sin^{-1}\left(\frac{1}{\sqrt{(4n^4)+1}}\right)$

upto infinity.

Sol$^n$: $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \dots \tan^{-1}\frac{1}{2n^2}$

$T_n = \cancel{\tan^{-1}\frac{1}{1+2n^2+1}}$ $\tan^{-1}\frac{2}{1+4n^2-1}$

$$T_n = \tan^{-1}\frac{2}{1+(2n-1)(2n+1)}$$

$$= \tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n-1)(2n+1)}$$

$$T_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$S_n = \cancel{\tan^{-1} 3} - \tan^{-1} 1 + \cancel{\tan^{-1} 5} - \cancel{\tan^{-1} 3} + \cdots + \tan^{-1}(2n+1) - \cancel{\tan^{-1}(2n-1)}$

$$S_n = \lim_{n\to\infty} \tan^{-1}(2n+1) - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Q. If $0 < a, b < 1$ and $\tan^{-1} a + \tan^{-1} b = \pi/4$, then the value of

$$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \cdots \text{ is}$$

Soln :-

$$\tan^{-1} a + \tan^{-1} b = \pi/4$$

$$\tan^{-1} \frac{a+b}{1-ab} = \frac{\pi}{4}$$

$$a+b = 1-ab$$

$$a+b+ab-1=0$$

$$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \cdots$$

$$\left(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{b^4}{4} - \cdots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \cdots\right)$$

$$= \ln(1+a) + \ln(1+b)$$

$$= \ln(1+a)(1+b)$$

$$= \ln(1+a+b+ab)$$

$$= \ln 2$$

$$= \log_e 2$$

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Q. $\sec^{-1}\left(\frac{1}{4}\sum_{K=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{K\pi}{2}\right)\cdot\sec\left(\frac{7\pi}{12}+\frac{(K+1)\pi}{2}\right)\right)$ in the interval $[-\pi/4, 3\pi/4]$ [JEE Advanced 2019]

Sol$^n$: $\sec^{-1}\left(\frac{1}{4}\sum_{K=0}^{10}\frac{\sin\left\{\left[\frac{7\pi}{12}+\frac{(K+1)\pi}{2}\right]-\left[\frac{7\pi}{12}+\frac{K\pi}{12}\right]\right\}}{\cos\left(\frac{7\pi}{12}+\frac{K\pi}{2}\right)\cos\left(\frac{7\pi}{12}+\frac{(K+1)\pi}{2}\right)}\right)$

$\cancel{\sec^{-1}}$ $\frac{7\pi}{12}+\frac{(K+1)\pi}{2}=A$

$\frac{7\pi}{12}+\frac{K\pi}{2}=B$

$$\frac{\sin A\cos B-\cos A\sin B}{\cos B\cdot\cos A}$$

$$\tan A-\tan B$$

$\sec^{-1}\left(\frac{1}{4}\sum_{K=0}^{10}\tan\left(\frac{7\pi}{12}+\frac{(K+1)\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{K\pi}{2}\right)\right)$

$\sec^{-1}\frac{1}{4}\left(\tan\left(\cancel{\frac{7\pi}{12}}+\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}\right)+\tan\left(\cancel{\frac{7\pi}{12}}+\pi\right)\right.$

$-\tan\left(\cancel{\frac{7\pi}{12}}+\frac{\pi}{2}\right)+\tan\left(\cancel{\frac{7\pi}{12}}+\frac{3\pi}{12}\right)-\tan\left(\cancel{\frac{7\pi}{12}}+\pi\right)$

$+\ldots+\tan\left(\cancel{\frac{7\pi}{12}}+5\pi\right)-\tan\left(\cancel{\frac{7\pi}{12}}+\frac{9\pi}{2}\right)$

$\left.+\tan\left(\frac{7\pi}{12}+\frac{11\pi}{2}\right)-\tan\left(\cancel{\frac{7\pi}{12}}+5\pi\right)\right)$

$=\sec^{-1}\left(\frac{1}{4}\left[\tan\frac{73\pi}{12}-\tan\frac{7\pi}{12}\right]\right)$

$$= \sec^{-1}\left(\frac{1}{4}\left[\tan\frac{\pi}{12} - \tan\frac{7\pi}{12}\right]\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(2-\sqrt{3}+2+\sqrt{3}\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\times 4\right)$$

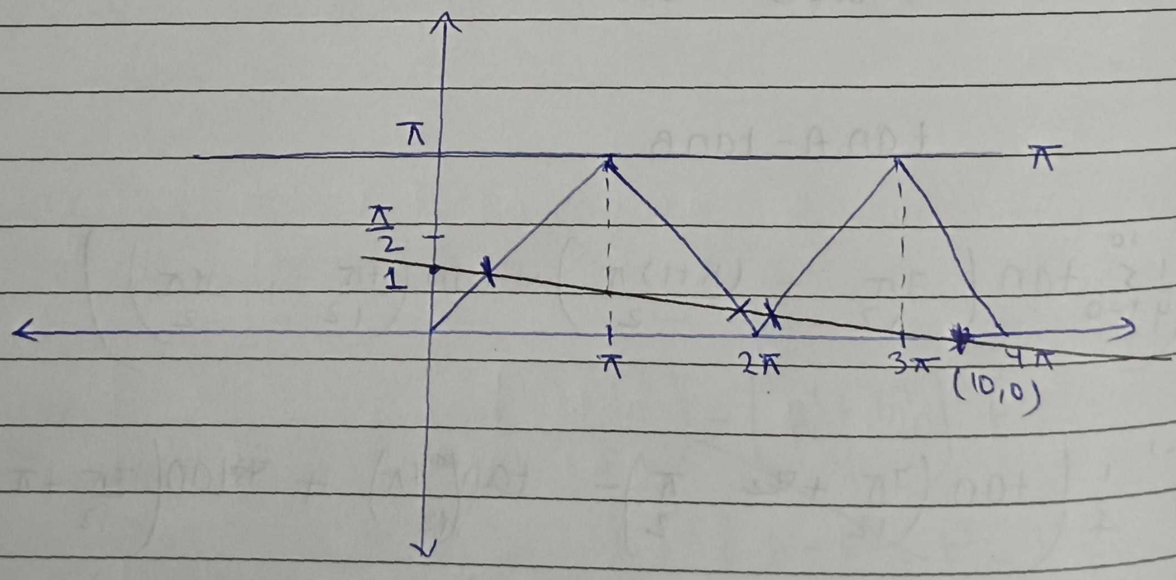
$$= \sec^{-1}(1)$$

$$= 0$$

Q. If $f:[0,4\pi]\to[0,\pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. Then the number of points $x\in[0,4\pi]$ satisfying the eqn $f(x) = \frac{10-x}{10}$ is

[JEE Advanced 2014]

Soln :-




$$f(x) = \frac{10-x}{10}$$

$$f(x) = 1 - \frac{x}{10}$$

No. of solutions = 3

# ~~Rem~~ Remember:-

1. $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2} \Rightarrow x = y = z = 1$

2. $\cos^{-1}x + \sin^{-1}y + \sin^{-1}z = 3\pi \Rightarrow x = y = z = -1$

3. $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ and
4. $\tan^{-1}1 + \tan^{-1}1/2 + \tan^{-1}1/3 = \frac{\pi}{2}$

Q. Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$, where $0 \le \cos^{-1}x \le \pi$ and $\sin^{-1}x \in [-\pi/2, \pi/2]$. [1981]

Sol$^n$:- $\cos[2\cos^{-1}x + \sin^{-1}x]$

~~cos( 2cos^-1 x +~~

$= \cos(\cos^{-1}x + \pi/2)$

$= \cos(-\sin^{-1}x)$

$= \cos\left(-\sin^{-1}\left(\frac{1}{5}\right)\right)$

$= \cos\left(-\cos^{-1}\frac{2\sqrt{6}}{5}\right)$

$=$

$= -\sin(\cos^{-1}x)$

$= -\sin(\cos^{-1}1/5)$

$= -\sin\left(\sin^{-1}\frac{2\sqrt{6}}{5}\right)$

$= -\frac{2\sqrt{6}}{5}$

Q. Considering only the principal values of the ITF, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}} + \frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2} + \tan^{-1}\frac{\sqrt{2}}{\pi}$$

[JEE Advanced 2022]

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Sol$^n$:-

$\cos x = \sqrt{\frac{2}{2+\pi^2}}$ $\frac{\to B}{\to H}$

$P^2 = 2+\pi^2-2$

$P = \pi$

$\tan x = \frac{\pi}{\sqrt{2}}$

$\sin x = \frac{2\sqrt{2}\pi}{2+\pi^2}$ $\frac{\to P}{\to H}$

Wrong approach

$B^2 = (2+\pi^2)^2 - (2\sqrt{2}\pi)^2$

$B^2 = 4+\pi^4+4\pi^2-8\pi^2$

$B^2 = \pi^4-4\pi^2+4$

$B^2 = \pi^4-2\pi^2-2\pi^2+4$

$B^2 = \pi^2(\pi^2-2)-2(\pi^2-2)$

$B = \pi^2-2$

$\therefore \tan x = \frac{2\sqrt{2}\pi}{\pi^2-2}$

$$\frac{3}{2}\tan^{-1}\frac{\pi}{\sqrt{2}} + \frac{1}{4}\tan^{-1}\frac{2\sqrt{2}\pi}{\pi^2-2} + \tan^{-1}\frac{\sqrt{2}}{\pi}$$

---

Sol$^n$:-

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+\frac{\pi^2}{2}}} + \frac{1}{4}\sin^{-1}\frac{\sqrt{2}\pi}{1+\frac{\pi^2}{2}} + \tan^{-1}\left(\frac{\sqrt{2}}{\pi}\right)$$

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}} + \frac{1}{4}\sin^{-1}\frac{\sqrt{2}\pi}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2} + \tan^{-1}\left(\frac{\sqrt{2}}{\pi}\right)$$

$\frac{3}{2}\ \frac{\pi}{\sqrt{2}} = \tan\theta$

$$\tan\frac{\pi}{4} < \cancel{\tan}\ \frac{\pi}{\sqrt{2}} < \tan\frac{\pi}{2}$$

$$\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \qquad 2\theta \in \left(\frac{\pi}{2}, \pi\right)$$

$$\cancel{\frac{3}{2}\theta}\ \frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+\tan^2\theta}} + \frac{1}{4}\sin^{-1}\frac{\sqrt{2}\pi}{1+\tan^2\theta} + \tan^{-1}(\cot\theta)$$

$$\frac{3}{2}\cos^{-1}(\cos\theta) + \frac{1}{4}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \tan^{-1}(\cot\theta)$$

$$\frac{3}{2}\theta + \frac{1}{4}\underbrace{\sin^{-1}(\sin 2\theta)}_{2\theta\in\left(\frac{\pi}{2},\pi\right)\to\pi-2\theta} + \frac{\pi}{2} - \cot^{-1}(\cot\theta)$$

$$= \frac{3}{2}\theta + \frac{1}{4}(\pi - 2\theta) + \frac{\pi}{2} - \theta$$

$$= \frac{3}{2}\theta - \frac{\theta}{2} - \theta + \frac{\pi}{4} + \frac{\pi}{2}$$

$$= \frac{3\pi}{4}$$

$$= \frac{3\times\cancel{22}^{11}}{7\times\cancel{4}_{2}}$$

$$= 2.36$$